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A point explosion in a cold exponential atmosphere. Part 2. Radiating flow

By DALLAS D. LAUMBACH† AND RONALD F. PROBSTEIN

Fluid Mechanics Laboratory, Department of Mechanical Engineering, Massachusetts Institute of Technology

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The problem considered is that of a strong shock propagating from a point energy source into a cold exponential atmosphere with radiative heat transfer in the flow behind the shock. The radiation mean free path is taken to be small compared to the shock radius, so that the shock may be treated as discontinuous and the radiative heat flux represented by the Rosseland diffusion approximation. The solution obtained is an approximate one based upon the 'local radiality' assumption and integral method previously utilized by Laumbach & Probstein for the case of adiabatic flow. The 'thin shock' concepts, which underlie the integral method, are extended to the present case of a radiating flow enabling an approximate integral of the differential energy equation to be obtained. A radiation parameter is developed, which provides an index as to when the effects of radiation may be neglected and the flow taken to be adiabatic. The physical interpretation of this parameter is that of the ratio of a characteristic radiation energy flux to a characteristic kinetic energy flux. When the value of this parameter is less than about 0·1, radiation effects may be neglected. It is shown that, when the radiation mean free path varies as a power of the temperature (T^n), where $n = -\frac{17}{6}$, the infinity of solutions for various polar angles can be transformed into two distinct solutions thereby essentially eliminating the parametric dependence on the polar angle and the atmospheric scale height. For fixed values of the radiation parameter the dependence on the explosion energy and the atmospheric density at the point of explosion is also eliminated. The results presented are for the mean free path–temperature variation indicated, but the technique of solution does not have this restriction, though for other temperature dependences some of the scaling advantages are sacrificed. The solution demonstrates the existence of an independence principle in which the flow becomes isothermal and independent of the detailed radiation mechanism when the radiation parameter becomes large. The limiting results of the present analysis for a uniform density atmosphere correspond quite well with the exact solutions of Elliott & Korobeinikov. The radiating far field behaviour of the descending shock is shown to approach that for adiabatic flow, and, consequently, the asymptotic adiabatic solution obtained by Raizer is an appropriate limit to the present solution. However, the asymptotic results for the ascending shock show that the radiating flow does not approach an adiabatic one but rather an isothermal one.

† Present address: Shell Development Company, Houston, Texas.

1. Introduction

In the first part of this work (Laumbach & Probstein 1969, hereinafter referred to as 'part 1'), a simplified integral theory was developed for calculating the flow field and shock behaviour resulting from a strong point explosion in an atmosphere whose density decreases exponentially with altitude. In part 1, the flow field behind the shock was assumed to be adiabatic. The purpose of the present paper is to extend that analysis to include the effects of radiation heat conduction behind the propagating shock.

The exact, self-similar solution for a heat-conducting flow behind a strong shock propagating in a uniform density atmosphere was first obtained by Korobeinikov (1957). Elliott (1960) published the solution to the same problem (apparently unaware of the earlier Russian work), but also pointed out the close approximation of the self-similar heat conduction term to that for radiation heat conduction (Rosseland diffusion approximation) and considered the problem in greater detail. In order to obtain a self-similar solution, Elliott found it necessary that the radiation mean free path λ vary as T^n, where T is the temperature and $n = -\frac{17}{6}$. This condition for self-similarity was first shown to be required with heat conduction in a spherically symmetric flow by Bam-Zelikovich (1949). For a very high temperature radiating flow, such as occurs immediately after an intense explosion, the heat flux is so large that the temperature profile is essentially flat and, consequently, the temperature gradient is zero. The solution for this case has been obtained by Korobeinikov (1956).

To the authors' knowledge no analytic solutions for an exponential atmosphere have been obtained with radiation heat conduction. One of the purposes of the analysis is to obtain an approximate analytic solution by the method of part 1 and to compare the limiting results for a uniform density atmosphere with the exact solutions mentioned.

2. Theory

As in part 1, the shock wave is assumed to be sufficiently strong that counter-pressure may be neglected and the strong shock relations applied. The gas is considered to be a calorically and thermally perfect one characterized by an appropriately selected adiabatic exponent γ and gas constant Γ. Body forces due to the earth's gravitational and magnetic fields and wind effects are neglected. Radiation pressure and energy are taken to be small in comparison with the material pressure and internal energy. The shock radius is assumed to be large in comparison with an appropriate characteristic radiation mean free path. These assumptions restrict the analysis to what is termed a 'low altitude' solution valid up to about 8–10 scale heights above the earth's surface. The atmosphere, considered to be initially at rest and cold, i.e. at zero temperature and pressure, is taken to have an exponential density distribution given by

$$\rho_0 = \rho_B e^{-h/\Delta}. \tag{1}$$

Here, Δ is the scale height of the atmosphere, taken to be a constant, ρ_0 is the atmospheric density, ρ_B is the density at the burst point, and h is the altitude measured positive upward from the point of explosion.

In figure 1 is shown a sketch of the shock envelope at a given time after the explosion, showing the polar co-ordinate system used in the analysis of part 1 and here. The basic simplification to be employed in the theory is what was termed in part 1 the 'local radiality' assumption. This assumption is valid when the flow is primarily radial, so that the streamlines from the origin can be taken as straight and the gradients in the θ-direction neglected. As the shock becomes increasingly asymmetric, the assumption of negligible gradients in the θ-direction becomes less satisfactory. However, it was pointed out in part 1 that the local radiality assumption is valid to about the same distances as is the strong shock assumption.

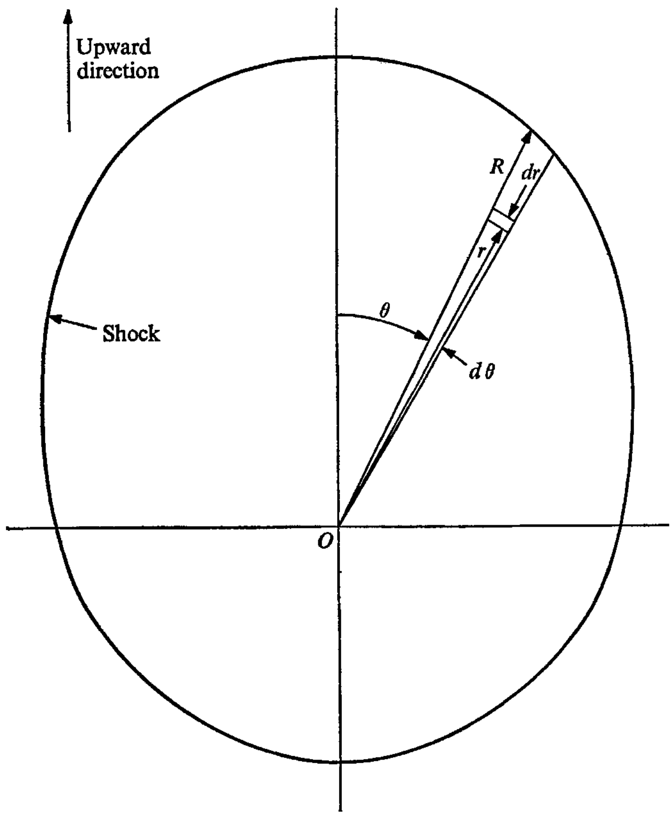

FIGURE 1. Flow geometry.

Under the assumptions noted the problem is axisymmetric about the vertical axis, with the origin in figure 1 the energy release point. Here r is the Eulerian co-ordinate of a fluid parcel of thickness dr, and $R(t;\theta)$ is the position of the shock front at a given polar angle θ. In the Lagrangian formulation used in part 1 and here, the exponential density distribution (1) becomes, in terms of the Lagrangian co-ordinate r_0,

$$\rho_0 = \rho_B \exp[-(r_0/\Delta)\cos\theta]. \tag{2}$$

Here, r_0 is defined as the co-ordinate r of a particular fluid particle at the burst time $t = 0$.

The continuity equation is, of course, unaltered by the presence of radiation, and with the local radiality assumption it may be written, for any polar angle, as

$$\rho_0 r_0^2 dr_0 = \rho r^2 dr. \tag{3}$$

With the neglect of radiation pressure the momentum equation is the same as that given in part 1, and, in Lagrangian co-ordinates, is

$$\frac{\partial^2 r}{\partial t^2} + \frac{r^2}{\rho_0 r_0^2}\frac{\partial p}{\partial r_0} = 0. \tag{4}$$

The energy equation is altered by the presence of a finite heat flux q, and, under the assumption that the gas is perfect and inviscid, it is given by

$$\frac{p}{\gamma-1}\frac{\partial}{\partial t}\left(\ln\frac{p}{\rho^\gamma}\right) = \frac{-\rho}{\rho_0 r_0^2}\frac{\partial}{\partial r_0}(r^2 q). \tag{5}$$

It can be shown (see for example, Vincenti & Kruger 1965) that if $|\partial B_\nu/\partial r|/\alpha_\nu B_\nu$ is small compared with unity, where B_ν is the Planck function and α_ν is the volumetric absorption coefficient, the Rosseland diffusion approximation may be used for calculating the radiation heat flux. This is equivalent to requiring that

$$Bu = \frac{\tilde{\lambda}}{R} \ll \frac{\tilde{T}}{\Delta\tilde{T}}. \tag{6}$$

Here, the tilde is used to denote appropriately defined characteristic values with λ a radiation mean free path, T a temperature, and ΔT a temperature difference over a distance of the order of the shock radius R. Since $\tilde{T}/\Delta\tilde{T}$, in general, is greater than one this condition is satisfied by our assumption that the Bouguer number Bu is considered to be small in comparison with unity.

The heat flux under the Rosseland diffusion approximation is given by

$$q = -\tfrac{16}{3}\lambda\sigma T^3\frac{\partial T}{\partial r}, \tag{7}$$

where σ is the Stefan–Boltzmann constant, and λ is now taken to be the Rosseland mean free path. In addition, for a perfect gas,

$$p = \rho\Gamma T, \tag{8}$$

where Γ is an appropriately selected gas constant.

Since the Bouguer number is assumed small the energy loss due to radiation 'leakage' out through the shock front is negligible. This is due to the fact that the 'radiation toe', which contains the preheated gas (and consequently the energy) ahead of the shock front, has a length of the same order as the radiation mean free path (see for example, Zel'dovich & Raizer 1966, pp. 526ff.). As a result, the heat flux ahead of the shock may be neglected. Under this condition, for a strong shock moving into a gas at rest with the radiation pressure and energy neglected, the conservation conditions across the shock yield (see Elliott 1960; or Zel'dovich & Raizer 1966, p. 532)

$$u_s = (1-\beta)\dot{R}, \tag{9a}$$

$$p_s = (1-\beta)\rho_0 \dot{R}^2, \tag{9b}$$

$$\frac{\rho_0}{\rho_s} \equiv \beta = \frac{\gamma-1}{\gamma+1}\left[1+\frac{2q_s}{p_s \dot{R}}\right]. \tag{9c}$$

Here, u is the particle velocity and $\dot{R}$ the shock velocity, with the subscript s denoting conditions immediately behind the shock.

We note that with the introduction of a finite heat flux behind the shock the density ratio β is not uniquely determined by γ alone, as for the usual strong shock conditions, but depends on time through the flow field solution itself. Eliminating the shock pressure p_s in (9c), we find

$$\beta = \frac{1}{\gamma+1}\left\{\gamma - \left(1 - \frac{2(\gamma+1)^2 q_s}{\rho_0 \dot{R}^3}\right)^{\frac{1}{2}}\right\}. \tag{10}$$

It may be seen that there are two cases for which β approaches the usual strong shock limit $(\gamma-1)/(\gamma+1)$. The first is for $q_s \rightarrow 0$ with $\dot{R}$ finite, while the second is for $\dot{R} \rightarrow \infty$ with q_s finite. Since the radiation mean free path is inversely proportional to the density, and since the density increases exponentially in the downward direction, it is clear that the radiation mean free path will tend to zero exponentially in the downward direction. Since the temperature gradient must remain finite, it follows from (7) that the heat flux at the shock approaches zero, whence $\beta \rightarrow (\gamma-1)/(\gamma+1)$ for the downward propagating shock. On the other hand, for the upward propagating shock at 'blowout', that is when $\dot{R} \rightarrow \infty$ (see part 1), β will also approach the usual strong shock limit.

In general, however, β depends on the shock location (or time). Its value can be determined from conservation of mass by integrating (3) from the burst point out to the shock front, with the result

$$\int_0^1 \left(\frac{r}{R}\right)^2 d\left(\frac{r}{R}\right) = \int_0^1 \left[\frac{\rho_0(r_0)}{\rho_s}\right]\left(\frac{\rho_s}{\rho}\right)\left(\frac{r_0}{R}\right)^2 d\left(\frac{r_0}{R}\right). \tag{11}$$

Here, ρ_s is the instantaneous density behind the shock which is held constant over the integration. Making use of (2) and the definition of $\beta \equiv \rho_0/\rho_s$ yields

$$\beta = \left[3\int_0^1 \left(\frac{\rho_s}{\rho}\right) \exp\left\{\left(\frac{R}{\Delta} - \frac{r_0}{\Delta}\right)\cos\theta\right\}\left(\frac{r_0}{R}\right)^2 d\left(\frac{r_0}{R}\right)\right]^{-1}. \tag{12}$$

The problem is now treated in a manner similar to the adiabatic flow case considered in part 1. The momentum equation is first written in the integral form,

$$p(r_0, t; \theta) - p_s(R; \theta) = \int_{r_0}^{R} \frac{1}{r^2}\frac{\partial^2 r}{\partial t^2}\rho_0 \bar{r}_0^2 d\bar{r}_0, \tag{13}$$

where p_s is the pressure immediately behind the shock front and $\bar{r}_0$ is a dummy variable. Since the heat flux ahead of the shock is negligible when the Bouguer number is small, the total energy of the flow field E remains essentially constant with time. Using (3) the energy equation may, for a given polar angle, be written in the integrated form,

$$\frac{E}{4\pi} = \int_0^R \frac{p}{\gamma-1} r^2 dr + \int_0^R \frac{1}{2}\left(\frac{\partial r}{\partial t}\right)^2 \rho_0 r_0^2 dr_0. \tag{14}$$

The integrals in (13) and (14) are now approximated in the same manner as for adiabatic flow, and use is once again made of the idea that most of the mass is located near the shock. It is to be noted from the strong shock condition (9*c*) that with a finite heat flux at the shock q_s, the mass is not concentrated at the shock to the same extent as in the adiabatic case. However, for γ close to 1, the preponderance of the mass is still located there. Consequently, a Taylor expansion for r in the Lagrangian co-ordinate r_0 is considered, with the expansion parameterized in the time t through the Taylor coefficients and R:

$$r(r_0, t) = R + \left.\frac{\partial r}{\partial r_0}\right|_R (r_0 - R) + \frac{1}{2}\left.\frac{\partial^2 r}{\partial r_0^2}\right|_R (r_0 - R)^2 + \ldots. \tag{15}$$

Only the first three terms in the expansion have been retained, since these are all that are required to obtain the expressions at the shock for r and its first two time derivatives, the velocity and the acceleration.

From the continuity equation (3) and the definition of β,

$$\left.\frac{\partial r}{\partial r_0}\right|_R = \left.\frac{\rho_0 r_0^2}{\rho r^2}\right|_R = \beta. \tag{16}$$

Using (16) and differentiating (15) with respect to t yields

$$r = R + \beta(r_0 - R) + \frac{1}{2}\left.\frac{\partial^2 r}{\partial r_0^2}\right|_R (r_0 - R)^2, \tag{17a}$$

$$\frac{\partial r}{\partial t} = (1-\beta)\dot{R} + \left(\frac{d\beta}{dt} - \left.\frac{\partial^2 r}{\partial r_0^2}\right|_R \dot{R}\right)(r_0 - R), \tag{17b}$$

$$\frac{\partial^2 r}{\partial t^2} = (1-\beta)\ddot{R} - 2\dot{R}\frac{d\beta}{dt} + \left.\frac{\partial^2 r}{\partial r_0^2}\right|_R \dot{R}^2. \tag{17c}$$

Evaluating the above relations at the shock, we find

$$r_s = R, \tag{18a}$$

$$\left(\frac{\partial r}{\partial t}\right)_s = (1-\beta)\dot{R}, \tag{18b}$$

$$\left(\frac{\partial^2 r}{\partial t^2}\right)_s = \frac{(1-\beta)^2}{1-2\beta}\ddot{R} + \frac{\beta(1-\beta)}{1-2\beta}\left\{\frac{2\dot{R}^2}{R}(1-\beta) - \frac{\dot{R}^2\cos\theta}{\Delta} - \frac{2\dot{R}^2}{1-\beta}\frac{d\beta}{dR} - \frac{3}{32}\frac{\beta(1-\beta)\rho_0\dot{R}^5}{\sigma\lambda_s T_s^4}\left[\frac{\gamma(\beta-1)+\beta+1}{\gamma-1}\right]\right\}. \tag{18c}$$

The detailed development leading to (18*c*), which is parameterized in θ, is somewhat subtle, and is given in the appendix.

The quantity $r^{-2}(\partial^2 r/\partial t^2)$ in the integrand of (13) is now approximated by its value at the shock given by (18). In this way, the dominant contribution to the integral is obtained. Replacing ρ_0 by (2) and integrating yields

$$p-p_s = \left(\frac{1-\beta}{1-2\beta}\right)\left(\frac{\Delta}{\cos\theta}\right)^2\frac{\rho_B}{\eta^3}\left\{2(1-\beta)\,\eta\ddot{\eta}-2\beta\eta\dot{\eta}^2-\frac{4\beta}{1-\beta}\frac{d\beta}{d\eta}\eta\dot{\eta}^2\right.$$
$$\left.-\frac{3\rho_B\eta\,e^{-\eta}}{16\sigma\lambda_s}\left(\frac{\cos\theta}{\Delta}\right)^4\frac{\Gamma^4[\gamma(\beta-1)+\beta+1]}{(\gamma-1)\beta^2(1-\beta)^3\dot{\eta}^3}+4\beta(1-\beta)\,\dot{\eta}^2\right\}$$
$$\times\left\{e^{-(r_0/R)\eta}\left[\left(\frac{r_0}{R}\right)^2\frac{\eta^2}{2}+\left(\frac{r_0}{R}\right)\eta+1\right]-e^{-\eta}\left[\frac{\eta^2}{2}+\eta+1\right]\right\}, \tag{19}$$

where the reduced variable η is defined by

$$\eta = \frac{R}{\Delta}\cos\theta, \tag{20}$$

and the pressure immediately behind the shock from (9*b*) and (2) is

$$p_s = (1-\beta)\,\rho_B\left(\frac{\Delta}{\cos\theta}\right)^2\dot{\eta}^2e^{-\eta}. \tag{21}$$

Use is now made of the condition that most of the mass is concentrated at the shock, which, to the order of the present approximation, implies that r is not a function of r_0, so that, for any $r \neq R$, the corresponding value of r_0 is zero (see part 1). Thus $p(r,t) = p(0,t)$, and the internal energy term of (14) is evaluated from (19) by setting $r_0 = 0$. The kinetic energy term in (14) is now approximated through the value of $(\partial r/\partial t)_s$ given by (18*b*). Replacing ρ_0 by (2), integrating and combining terms leads to an ordinary, second-order differential equation for η as a function of time:

$$f(\eta)\,\ddot{\eta}+g(\eta)\,\dot{\eta}^2-h(\eta)\left(\frac{\cos\theta}{\Delta}\right)^4\frac{1}{\lambda_s\dot{\eta}^3} = \frac{E}{4\pi\rho_B}\left(\frac{\cos\theta}{\Delta}\right)^5. \tag{22}$$

Here $f(\eta)$, $g(\eta)$ and $h(\eta)$ are defined by

$$f(\eta) = \frac{2}{3}\frac{(1-\beta)^2\eta}{(1-2\beta)(\gamma-1)}\left[1-e^{-\eta}\left(\frac{\eta^2}{2}+\eta+1\right)\right], \tag{23a}$$

$$g(\eta) = \frac{(1-\beta)\,\eta^3e^{-\eta}}{3(\gamma-1)}+\left[\frac{2\beta}{\eta}-\frac{\beta}{1-\beta}-\frac{2\beta}{(1-\beta)^2}\frac{d\beta}{d\eta}+\frac{3}{2}\frac{(1-2\beta)(\gamma-1)}{\eta}\right]f(\eta), \tag{23b}$$

$$h(\eta) = \frac{1}{16}\frac{\eta\,e^{-\eta}[\gamma(\beta-1)+\beta+1]}{(1-2\beta)(\gamma-1)^2\beta^2(1-\beta)^2}. \tag{23c}$$

As in the adiabatic case we introduce the reduced time,

$$t^* = t\left[\frac{E|\cos^5\theta|}{4\pi\rho_B\Delta^5}\right]^{\frac{1}{2}}. \tag{24}$$

In terms of this reduced time, (22) transforms to

$$f(\eta)\,\eta''+g(\eta)\,\eta'^2-h(\eta)\frac{\rho_B\Gamma^4}{\sigma\lambda_s\eta'^3}\left(\frac{4\pi\rho_B}{E}\right)^{\frac{5}{2}}\left(\frac{\Delta}{|\cos\theta|}\right)^{\frac{17}{2}} = 1 \quad \text{for} \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi, \tag{25a}$$

$$f(\eta)\,\eta''+g(\eta)\,\eta'^2-h(\eta)\frac{\rho_B\Gamma^4}{\sigma\lambda_s\eta'^3}\left(\frac{4\pi\rho_B}{E}\right)^{\frac{5}{2}}\left(\frac{\Delta}{|\cos\theta|}\right)^{\frac{17}{2}} = -1 \quad \text{for} \quad \tfrac{1}{2}\pi \leqslant \theta \leqslant \pi, \tag{25b}$$

where the prime is used to denote differentiation with respect to t^*.

From (25) it is clear that, if the parametric dependence of the solution on θ is to be eliminated, this must be accomplished through the temperature dependence of λ, and, further, that this must be a power law relation. Therefore, the dependence of λ on the thermodynamic variables ρ and T is taken to be

$$\lambda = a\Phi\left(\frac{\rho}{\rho_g}\right)T^n, \tag{26}$$

where ρ_g is standard density and Φ is an arbitrary, dimensionless function. Since $\Gamma T_s \propto \dot{R}^2 \propto \eta'^2|\cos^3\theta|$, to achieve this elimination n must be given by

$$n = -\tfrac{17}{6}. \tag{27}$$

This coincides with the temperature variation of λ required in order for a self-similar solution to exist in the uniform density case (see for example, Elliott 1960). The conditions for the existence of a self-similar solution with any type of heat conduction in a spherically symmetric flow were first formulated by Bam-Zelikovich (1949) (see also Sedov 1957, p. 232).

Radiation mean free path data (see, for example, Armstrong *et al.* 1961) show that (27) is a realistic approximation to the temperature variation of λ over the range from 6000 to 60,000 °K. Over this temperature range the density variation is given quite closely by

$$\Phi\left(\frac{\rho}{\rho_g}\right) = \frac{\rho_g}{\rho}. \tag{28}$$

In view of the scaling advantages and the relatively large temperature range where this is valid, the dependence of λ on the thermodynamic variables is taken to be

$$\lambda = a\left(\frac{\rho_g}{\rho}\right)T^{-\frac{17}{6}}. \tag{29}$$

Substituting the above value of λ into (25) yields

$$f(\eta)\eta'' + g(\eta)\eta'^2 - H(\eta)(\eta'^2)^{\frac{4}{3}} = 1 \quad (0 \leqslant \theta \leqslant \tfrac{1}{2}\pi), \tag{30a}$$

$$f(\eta)\eta'' + g(\eta)\eta'^2 + H(\eta)(\eta'^2)^{\frac{4}{3}} = -1 \quad (\tfrac{1}{2}\pi \leqslant \theta \leqslant \pi), \tag{30b}$$

where
$$H(\eta) = \frac{e^{-2\eta}[\gamma(\beta-1)+\beta+1]}{2\kappa(1-\beta)^{\frac{7}{6}}\beta^{\frac{1}{6}}(\gamma-1)}f(\eta), \tag{31}$$

with the 'radiation parameter' κ defined by

$$\kappa = \frac{16}{3}\left(\frac{\rho_g}{\rho_B}\right)\frac{\sigma a}{\rho_B\Gamma^{\frac{7}{6}}}\left(\frac{4\pi\rho_B}{E}\right)^{\frac{1}{3}}. \tag{32}$$

It may be seen that through the adoption of (29) and the introduction of the length scale $\Delta/\cos\theta$ and the time scale $(4\pi\rho_B\Delta^5/E|\cos^5\theta|)^{\frac{1}{2}}$ it is possible to scale all motions of the ascending shock from a single solution of (30a) and all motions of the descending shock from a single solution of (30b). Further, either solution can be scaled for arbitrary values of Δ, while for fixed values of the radiation parameter κ either solution can also be scaled for arbitrary values of E and ρ_B.

The effect of radiation heat transfer is seen to enter only through the parameter κ appearing in the function $H(\eta)$. This parameter is equivalent to the ratio of the Bouguer number to the Boltzmann number (for definitions see, for example, Goulard 1964), and can be physically interpreted as the ratio of a characteristic radiation energy flux to a characteristic kinetic energy flux. It is analogous to the quantity A introduced by Korobeinikov (1957) and the quantity K introduced by Elliott (1960). In interpreting his parameter K, Elliott concluded that it increased with increasing energy yield. However, it is clear that, on the contrary, the radiation parameter increases for decreasing energy yield. Furthermore, the effect on κ of a change in burst point density is seen to be much greater than that which would result from the same relative change in E.

Since (30) are autonomous, they can be recast with η as the independent variable to yield two first-order equations in η'^2:

$$\frac{f(\eta)}{2}\frac{d(\eta'^2)}{d\eta}+g(\eta)\,(\eta'^2)-H(\eta)\,(\eta'^2)^{\frac{4}{3}} = 1 \quad \text{for} \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi, \tag{33a}$$

$$\frac{f(\eta)}{2}\frac{d(\eta'^2)}{d\eta}+g(\eta)\,(\eta'^2)+H(\eta)\,(\eta'^2)^{\frac{4}{3}} = -1 \quad \text{for} \quad \tfrac{1}{2}\pi \leqslant \theta \leqslant \pi. \tag{33b}$$

These equations can be integrated by straightforward numerical methods for first-order equations (e.g. Runge–Kutta technique), by making use of the expression (12) for β, which can be evaluated once the density distribution has been determined. The appropriate initial conditions at $\eta = 0$ are determined from the uniform density solution discussed in § 3. Once the integration has been carried out, the behaviour of $\eta = \eta(t^*)$ can be found from the relation,

$$t^* = \int_0^{\eta} \frac{d\overline{\eta}}{\overline{\eta}'}. \tag{34}$$

The pressure distribution is given from (19), (24), (29) and (32) by

$$\frac{p}{p_s} = 1+\left\{\frac{2(1-\beta)}{1-2\beta}\,\frac{\eta''}{\eta^2\eta'^2}+\frac{4\beta(1-\beta)}{1-2\beta}\,\frac{1}{\eta^3}-\frac{4\beta}{(1-2\beta)\,(1-\beta)}\,\frac{d\beta}{d\eta}\,\frac{1}{\eta^2}-\frac{2\beta}{1-2\beta}\,\frac{1}{\eta^2} - \frac{e^{-2\eta}\,|\eta\eta'^{\frac{2}{3}}|\,[\gamma(\beta-1)+\beta+1]}{(1-2\beta)\,(1-\beta)^{\frac{1}{6}}\,\beta^{\frac{1}{6}}\kappa(\gamma-1)\,\eta^3}\right\}\left\{e^{(1-r_0/R)\eta}\left[\left(\frac{r_0}{R}\right)^2\frac{\eta^2}{2}+\left(\frac{r_0}{R}\right)\eta+1\right]-\left[\frac{\eta^2}{2}+\eta+1\right]\right\}, \tag{35}$$

with p_s obtainable from (21).

To determine the density distribution the energy equation (5) must be integrated. We note that if, say, the two variables p and q were known as a function of r_0 and t, then, in principle, the equation could be integrated for ρ. Since the pressure distribution has been determined, this requires that q be known. The heat flux is however not known in advance and even if it were the approach outlined would at best be difficult. Once again, therefore, we resort to an approximate integral method which makes use of the idea that the mass is concentrated at the shock.

The pressure is first of all expanded in a Taylor series about $r_0 = 0$. Since we have chosen not to consider the time t explicitly, the expansion is written with t parameterized in the Taylor coefficients as

$$p(r_0, t) = p_b(t)+\left.\frac{\partial p}{\partial r_0}\right|_b r_0+\left.\frac{\partial^2 p}{\partial r_0^2}\right|_b \frac{r_0^2}{2}+\ldots, \tag{36}$$

with similar expressions written for the temperature and density. Here, the subscript b refers to conditions at the burst point where r and r_0 are both zero. A similar expansion for q gives

$$q(r_0, t) = \frac{\partial q}{\partial r_0}\bigg|_b r_0 + \frac{\partial^2 q}{\partial r_0^2}\bigg|_b \frac{r_0^2}{2} + \ldots, \tag{37}$$

with $q_b(t) = 0$, since by symmetry the temperature gradient is zero at the burst point.

As $\gamma \to 1$, and the mass becomes more concentrated at the shock, r_0 approaches zero everywhere except at the shock front, and consequently p, ρ and T become invariant with r except within a relatively thin layer near the shock. However, this limit must be treated carefully, since with radiation there is a finite temperature with a zero temperature gradient at the burst point, as opposed to the adiabatic case, where both of these quantities are infinite at the burst point. As $\gamma \to 1$, the flow becomes adiabatic (the temperature of a fluid particle remains constant with time), as may be seen from the energy equation (5), and consequently this limit is approached non-uniformly. Therefore, as γ gets too close to 1 or κ gets too close to 0, such that the flow becomes adiabatic, the approximation may get progressively worse instead of better. This is easily seen by multiplying (5) through by $\gamma - 1$ and noting from (7), (29) and (32) that q is proportional to κ, a result we have already noted in pointing out that κ is a measure of the radiative heat flux relative to the kinetic energy flux. It is therefore clear that either of these limiting values of κ or γ will cause the flow to become adiabatic, thereby invalidating the Taylor expansion about the burst point.

A more obvious restriction on the analysis is the one imposed by the requirement that the mass be concentrated at the shock. Consequently, when the shock density ratio β is too large, the errors introduced by the approximations may be unacceptable. The behaviour of β with respect to γ and κ may be qualitatively estimated from (9b) and (9c) by once again noting that q is proportional to κ.

In solving the energy equation (5), it is found convenient to do so in Eulerian variables. We therefore rewrite it in the form,

$$\frac{\partial(\rho q)}{\partial r} = -\frac{\rho p}{\gamma - 1}\frac{D}{Dt}\left(\ln\frac{p}{\rho^\gamma}\right) - \frac{2\rho q}{r} + q\frac{\partial \rho}{\partial r}. \tag{38}$$

As a first approximation to the functions integrated over r, we use the idea that most of the mass is concentrated at the shock, and choose to consider p and ρ appearing on the right-hand side of (38) to be independent of r. This approximation applies away from the shock layer for small r_0/R. This is exactly the same procedure which was carried out in evaluating the first term of (14). From this integration an expression for ρq can be obtained from which the temperature distribution can be determined. Knowing the temperature distribution and the pressure distribution, the density distribution can then be obtained from the equation of state (8).

Before carrying out the integration, however, it is necessary to approximate on the right-hand side of (38) the variation of q with r. When ρ is independent of r, it can be seen from the continuity equation (3) that r_0 varies linearly with r, since r_0 is small and therefore ρ_0 is essentially constant. Thus, as the mass becomes concentrated at the shock, an appropriate tentative first approximation is to take q as a linear function of r in carrying out the integration of (38). It must be emphasized that it is not assumed that q varies linearly with r in the problem, but only that to determine the dominant contribution to the integral on the right-hand side this is an appropriate first approximation for small r_0/R.

Under the approximations noted, the right-hand side of (38) becomes a function of t alone and, on integrating from any point r to the shock, leads to a relation of the form $\rho_s q_s - \rho q = F(t)(R-r)$. Clearly, conditions at the shock are automatically satisfied. However, at the origin, where $r \to 0$, this relation will in general give a finite heat flux, so that the symmetry condition of $q = 0$ (i.e. $\partial T/\partial r = 0$) cannot be satisfied. To overcome this difficulty, we introduce into (38) a function of r or 'weighting factor' containing an arbitrary parameter. The function's behaviour is such that, while still making use of the above approximations, the boundary conditions can be satisfied. We choose r^m for the function with which to weight ρq, and rewrite (38) in the form,

$$r^m \frac{\partial(\rho q/r^m)}{\partial r} = -\frac{p\rho}{\gamma-1}\frac{D}{Dt}\left(\ln\frac{p}{\rho^\gamma}\right) - \frac{m+2}{r}\rho q + q\frac{\partial \rho}{\partial r}, \tag{39}$$

with the requirement that the parameter $m > 1$.

Integrating (39)

$$-\int_{\rho q/r^m}^{\rho_s q_s/R^m} d\left(\frac{\bar{\rho}\bar{q}}{\bar{r}^m}\right) = \left[\frac{\rho_b p_b}{\gamma-1}\frac{d}{dt}\left(\ln\frac{p}{\rho^\gamma}\right)_b + (m+2)\rho_b \frac{q}{r}\bigg|_b\right]\int_r^R \frac{d\bar{r}}{\bar{r}^m}, \tag{40}$$

where the barred quantities are dummy variables, or

$$\rho q = \rho_s q_s\left(\frac{r}{R}\right)^m + \left[\frac{\rho_b p_b R}{\gamma-1}\frac{d}{dt}\left(\ln\frac{p}{\rho^\gamma}\right)_b + (m+2)\rho_b\frac{q}{r}\bigg|_b R\right]\left(\frac{1}{m-1}\right)\left[\frac{r}{R} - \left(\frac{r}{R}\right)^m\right]. \tag{41}$$

Solving for $(q/r)_b$, and taking the limit as $r \to 0$,

$$\frac{q}{r}\bigg|_b = -\frac{1}{3}\left[\frac{p_b}{\gamma-1}\frac{d}{dt}\left(\ln\frac{p}{\rho^\gamma}\right)_b\right]. \tag{42}$$

Substituting the above result into (41), we have finally

$$\rho q = \rho_s q_s\left(\frac{r}{R}\right)^m + \frac{1}{3}\frac{\rho_b p_b R}{\gamma-1}\frac{d}{dt}\left(\ln\frac{p}{\rho^\gamma}\right)_b\left[\left(\frac{r}{R}\right)^m - \frac{r}{R}\right]. \tag{43}$$

Equation (43) (or (41)) is seen to give the correct value of ρq at the shock and at the burst point, and is thus essentially analogous to the type of result obtained from an interior weighted residual method, where the boundary conditions are satisfied exactly and the differential equation is approximated within the interior region (see, for example, Ames 1965). The free parameter m corresponds to the usual undetermined parameter in the method of weighted residuals. Ordinarily, the undetermined parameter is selected by requiring that the integral of the error, weighted by some arbitrary weighting function, be zero. Instead of arbitrarily picking such a function to determine m, the role which it plays in (43) is considered. First, it is noted that there is a linear relation between ρq

and r/R for small r/R which is appropriate. This linear relation is then corrected to its proper value at the shock by a power law behaviour in $(r/R)^m$, with the correction significant within a layer of thickness of the order of R/m. This correction must occur in the régime where the density gradient is large, i.e. within the shock layer. Hence, m must be of the same order of magnitude as the inverse of the shock density ratio, and we therefore choose to set the two equal, such that

$$m = 1/\beta. \tag{44}$$

In carrying out the numerical solutions, it was found that the results were quite insensitive to perturbations of m about $1/\beta$.

To obtain the temperature distribution, we now substitute (7) and (29) into (43), and on integrating obtain, with $m = 1/\beta$,

$$\left(\frac{T}{T_s}\right)^{\frac{7}{6}} = 1+\frac{7}{6\kappa\rho_B^{\frac{2}{3}}(\Gamma T_s)^{\frac{7}{6}}}\left(\frac{4\pi\rho_B}{E}\right)^{\frac{1}{3}}\left\{\left[\rho_s q_s R+\frac{1}{3}\frac{\rho_b p_b R^2}{\gamma-1}\frac{d}{dt}\left(\ln\frac{p}{\rho^\gamma}\right)_b\right]\right.$$
$$\left.\times\left(\frac{\beta}{1+\beta}\right)\left[1-\left(\frac{r}{R}\right)^{(1+\beta)/\beta}\right]-\frac{1}{6}\frac{\rho_b p_b R^2}{\gamma-1}\frac{d}{dt}\left(\ln\frac{p}{\rho^\gamma}\right)_b\left[1-\left(\frac{r}{R}\right)^2\right]\right\}, \tag{45}$$

where use has been made of the definition (32). It should be noted that the above results are restricted to the temperature dependence of λ given by (29). However, the approximate technique of integration and the result for ρq given by (43) are not. In principle, any other analytic relation for λ as a function of temperature could be used, however, the scaling advantages introduced through a T^n dependence with $n = -\frac{17}{6}$ would be lost.

Utilizing (2), (8), (9*b*), (9*c*), (20) and (24) reduces (45) to

$$\left(\frac{T}{T_s}\right)^{\frac{7}{6}} = 1+\frac{7}{6}\frac{e^{-2\eta}|\eta'^{\frac{2}{3}}\eta|}{\kappa(1-\beta)^{\frac{1}{3}}\beta^{\frac{13}{6}}(\gamma-1)}\left\{\left[\frac{\gamma(\beta-1)+\beta+1}{2}+\frac{1}{3}\left(\frac{\rho_b}{\rho_s}\right)\left(\frac{p_b}{p_s}\right)\frac{\eta}{\eta'}\frac{d}{dt^*}\left(\ln\frac{p}{\rho^\gamma}\right)_b\right]\right.$$
$$\left.\times\left(\frac{\beta}{1+\beta}\right)\left[1-\left(\frac{r}{R}\right)^{(1+\beta)/\beta}\right]-\frac{1}{6}\left(\frac{\rho_b}{\rho_s}\right)\left(\frac{p_b}{p_s}\right)\frac{\eta}{\eta'}\frac{d}{dt^*}\left(\ln\frac{p}{\rho^\gamma}\right)_b\left[1-\left(\frac{r}{R}\right)^2\right]\right\}. \tag{46}$$

With this expression for the temperature distribution, it is now possible to evaluate the density distribution by making use of the gas law (8) and the pressure distribution (35). The resulting expression for the density distribution is, however, a function of β and therefore the determination of β through (12) must be found by an iterative method. Once β has been determined, the basic equations (33) and (34) can then be integrated.

With β determined as above, the density distribution is given explicitly, and the relation between r and r_0 can be found by integrating the continuity equation (3), which, in combination with (2) and the definition $\beta = \rho_0/\rho_s$, yields

$$\frac{r}{R} = \left[3\beta\int_0^{r_0/R}\left(\frac{\rho_s}{\rho}\right)\exp\left[\left(1-\frac{\bar{r}_0}{R}\right)\eta\right]\left(\frac{\bar{r}_0}{R}\right)^2 d\left(\frac{\bar{r}_0}{R}\right)\right]^{\frac{1}{3}}. \tag{47}$$

Since $u = \partial r/\partial t$, the velocity distribution may then be found by differentiating this expression to give

$$\frac{u}{u_s} = -\left(\frac{R}{r}\right)^2\frac{\beta}{1-\beta}\int_0^{r_0/R}\left(\frac{\rho_s}{\rho}\right)\left[\frac{R}{\dot{R}}\frac{1}{\rho}\frac{\partial\rho}{\partial t}\right]\exp\left[\left(1-\frac{r_0}{R}\right)\eta\right]\left(\frac{r_0}{R}\right)^2 d\left(\frac{r_0}{R}\right). \tag{48}$$

For calculational purposes, it is convenient to introduce the reduced Lagrangian co-ordinate

$$\xi = r_0/R. \tag{49}$$

Making use of the definition of η given by (20) and the chain rule for partial derivatives, the following expression for the velocity is obtained:

$$\frac{u}{u_s} = \left(\frac{R}{r}\right)^2 \frac{\beta}{1-\beta}\int_0^{r_0/R} \left(\frac{\rho_s}{\rho}\right)\left[\frac{\xi}{\rho}\left(\frac{\partial \rho}{\partial \xi}\right)_{t*} - \frac{\eta}{\eta'}\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t^*}\right)_\xi\right] e^{(1-\xi)\eta}\xi^2 d\xi. \tag{50}$$

From (47) the pressure, density, and velocity can now be found as a function of the Eulerian co-ordinate r.

3. Uniform density atmosphere

Since no solutions for a radiating point explosion in an exponential atmosphere are available, with which to compare the present results, we consider the limiting case of a uniform density atmosphere for which there are other solutions. The exact self-similar solution for heat-conducting flow was first presented by Korobeinikov (1957), with numerical calculations for what corresponds to one value of the present radiation parameter. The solution was also obtained by Elliott (1960), who noted that the form of the heat conduction term required for self-similarity is a good approximation to that for radiation diffusion over a relatively large temperature range. He published his results in considerably more detail, along with numerical calculations for what corresponds to three values of the present radiation parameter. In terms of the present analysis, this case is given by the limit $\eta = R\cos\theta/\Delta \to 0$.

In the limit $\eta \to 0$, the basic differential equations governing the shock propagation reduce to the single equation,

$$\ddot{R} + \frac{B}{R}\dot{R}^2 - \frac{[\gamma(\beta-1)+\beta+1]}{2(\gamma-1)\kappa\beta^{\frac{1}{6}}(1-\beta)^{\frac{7}{6}}}\left(\frac{4\pi\rho_B}{E}\right)^{\frac{1}{3}}\dot{R}^{\frac{8}{3}} = \frac{9(1-2\beta)(\gamma-1)}{(1-\beta)^2}\frac{E}{4\pi\rho_B}\frac{1}{R^4}, \tag{51}$$

where the equation is here written in dimensional variables, and where

$$B = \frac{3+\beta-10\beta^2+3\gamma(1-3\beta+2\beta^2)}{2(1-\beta)}. \tag{52}$$

The first integration of (51), subject to the condition that $\dot{R} \to 0$ as $R \to \infty$, gives

$$\dot{R}^2 = \frac{A}{R^3}\left(\frac{E}{4\pi\rho_B}\right). \tag{53}$$

Here, A is a constant given by the expression,

$$(B-\tfrac{3}{2})A - \left[\frac{\gamma(\beta-1)+\beta+1}{2(\gamma-1)\kappa\beta^{\frac{1}{6}}(1-\beta)^{\frac{7}{6}}}\right]A^{\frac{4}{3}} = \frac{9(1-2\beta)(\gamma-1)}{(1-\beta)^2}, \tag{54}$$

which can be solved by standard iterative techniques. Integrating (53) with $R = 0$ at $t = 0$ yields

$$R = \left(\frac{25AE}{16\pi\rho_B}\right)^{\frac{1}{5}} t^{\frac{2}{5}}. \tag{55}$$

Equation (55) yields values for R which are larger than the exact values for $\gamma = 1{\cdot}2$ given by Elliott (1960) by 1·0 % for $\kappa = 0{\cdot}0075$, 5·5 % for $\kappa = 0{\cdot}60$ and 14·3 % for $\kappa = 5{\cdot}25$. These values of κ correspond to values of Elliott's $\bar{K}$ of 0·5, 10 and 100, respectively. The increasing discrepancy between the values given by the present analysis and the exact values can be attributed to the fact that for increasing κ, there is a diminishing proportion of the mass concentrated at the shock front as evidenced by the increasing value of the shock density ratio β. This has a direct effect on the approximation to the internal energy term in (14) thereby underestimating it with a consequent overprediction of the kinetic energy of the flow.

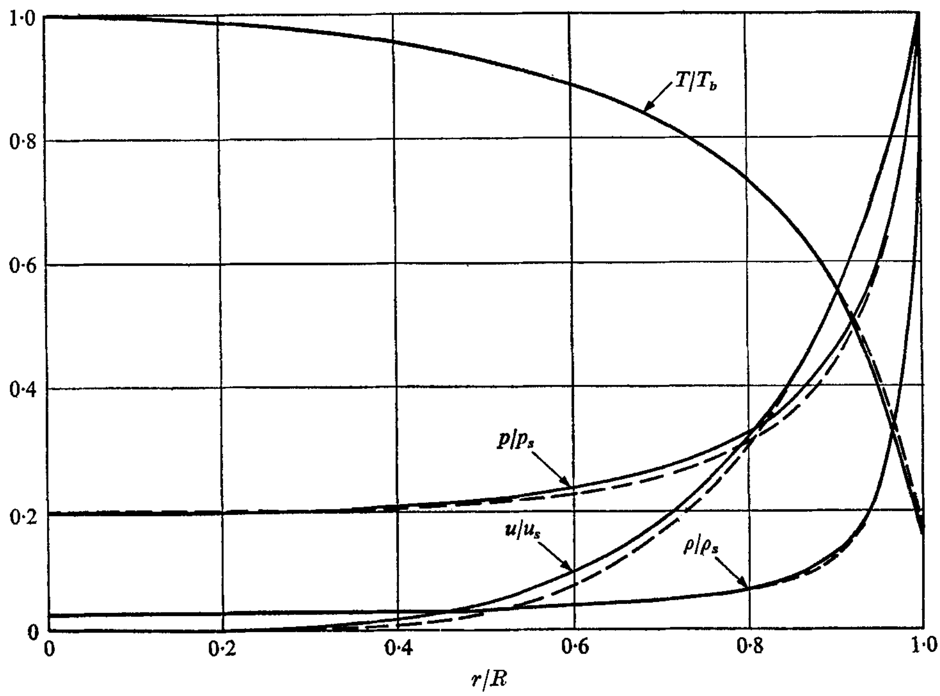

FIGURE 2. Flow variable distributions in Eulerian co-ordinates for uniform density atmosphere with $\kappa = 0{\cdot}60$ and $\gamma = 1{\cdot}2$. ——, present analysis; — — —, Elliott (1960) exact.

A comparison of the predictions for pressure, density, temperature and particle velocity behind the shock front for $\gamma = 1{\cdot}2$ and $\kappa = 0{\cdot}60$ are shown in figure 2. Here u is the particle velocity $(\partial r/\partial t)_{r_0}$, and the flow variables are reduced by their values at the shock front, with the exception of the temperature, which is reduced by its value at the burst point. These quantities are plotted as a function of the Eulerian co-ordinate r, which is reduced by the shock position R. The results of the present analysis are seen to be in excellent agreement with the exact self-similar solution. The results for $\gamma = 1{\cdot}2$ and $\kappa = 0{\cdot}0075$ are compared with the exact solution in figure 3, and once again the comparison is quite close, but not to the same degree as for $\kappa = 0{\cdot}60$. This difference is due to the smallness

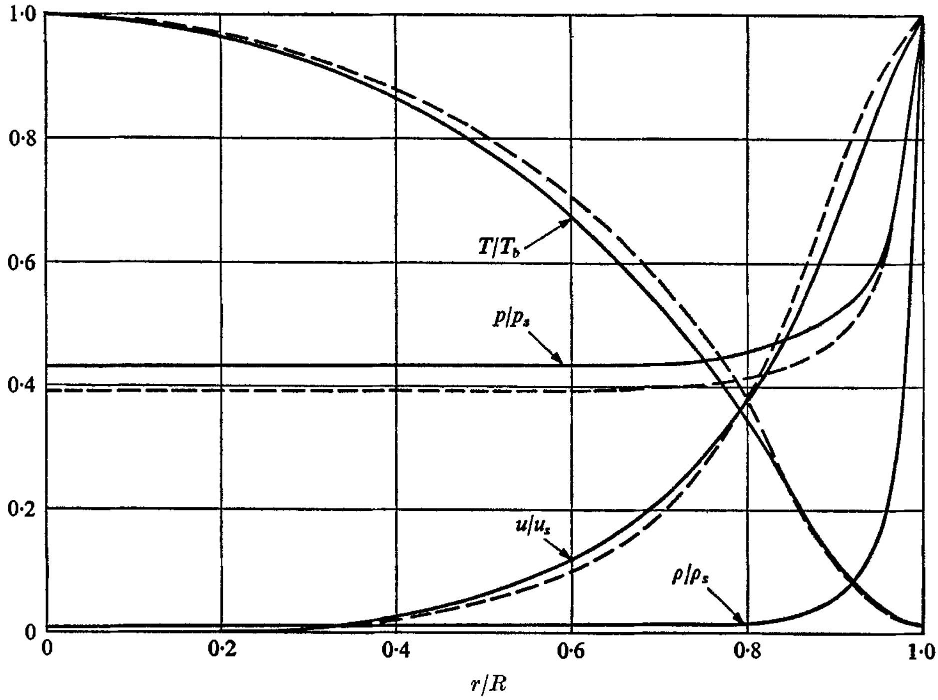

FIGURE 3. Flow variable distributions in Eulerian co-ordinates for uniform density atmosphere with $\kappa = 0{\cdot}0075$ and $\gamma = 1{\cdot}2$. ——, present analysis; — — —, Elliott (1960) exact.

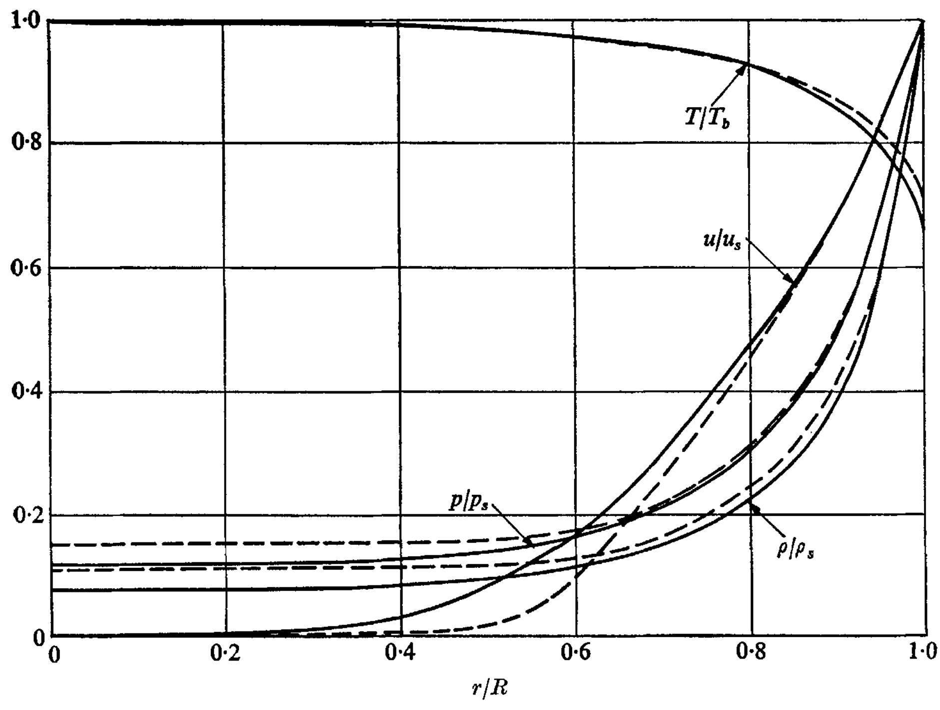

FIGURE 4. Flow variable distributions in Eulerian co-ordinates for uniform density atmosphere with $\kappa = 5{\cdot}25$ and $\gamma = 1{\cdot}2$. ——, present analysis; — — —, Elliott (1960) exact.

of κ and the consequent approach of the solution to that for adiabatic flow, thereby diminishing the validity of the approximations used in arriving at (43). The results for $\kappa = 5{\cdot}25$ with $\gamma = 1{\cdot}2$ are given in figure 4. The comparison with the exact solution in this case is the least satisfying; however, the behaviour of the flow variables is still predicted quite well. The discrepancy in this case is, of course, due to the lack of mass concentration at the shock.

It is clear from an examination of figures 2–4 that the departure of the flow variable distributions from those for adiabatic flow becomes significant for κ of the order of 0·1. It is of interest to note that the value of κ corresponding to an energy source of 20 KT released near the earth's surface is about 0·01. In view of the slow variation of κ with the total energy E (see (32)), it is clear that, for surface bursts (and in fact for altitudes up to about 3Δ), the flow may be considered as adiabatic.

4. Radiation parameter independence principle

It is easily seen from (30) and (31) that, when κ is large, the shock wave behaviour becomes independent of the value of κ. Further, the flow variables also exhibit an independence, since in this limit the temperature becomes uniform throughout the flow field as may be seen from (46). Consequently, the limit $\kappa \to \infty$ corresponds to isothermal flow behind the shock front, and there exists what we choose to call a 'radiation parameter independence principle'.

For the upward propagating shock the independence principle can be applied for values of κ in excess of about 10. For the downward propagating shock, however, κ must have a larger value in order to offset the eventual effect of the $e^{-2\eta}$ factor in both (31) and (46). It is quite clear that the requisite value of κ for applying this principle to the downward propagating shock is dependent upon the distance to which one wishes to extend the analysis. For a distance of 2Δ at a polar angle of π, κ should be in excess of about 100, for example.

The exact, self-similar solution in a uniform density atmosphere for the case when the independence principle applies (i.e. the isothermal limit) was obtained by Korobeinikov (1956). In this limit (54) reduces to

$$A = \frac{18(1-2\beta)(\gamma-1)}{[4\beta-10\beta^2+3\gamma(1-3\beta+2\beta^2)](1-\beta)}. \tag{56}$$

This expression gives a value for R in (55), which for $\gamma = 1{\cdot}4$ is in excess of Korobeinikov's value by 12·2 %. This difference can again be directly attributed to the small proportion of the mass which is concentrated at the shock front.

In this limit, i.e. $\eta \to 0$ and $\kappa \to \infty$, the pressure distribution in (35) reduces to

$$\frac{p}{p_s} = 1 + \frac{(1-\beta)(4\beta-3)}{6(1-2\beta)}\left[1-\left(\frac{r_0}{R}\right)^3\right], \tag{57}$$

and, since the flow is isothermal,

$$\frac{p}{p_s} = \frac{\rho}{\rho_s}. \tag{58}$$

Substituting (57) and (58) into the expression for β (12) yields

$$\frac{3-5\beta-4\beta^2}{6(1-2\beta)} = \exp\left[\frac{(1-\beta)(4\beta-3)}{6\beta(1-2\beta)}\right]. \tag{59}$$

From this result it is seen that the density ratio across the shock is independent of the value of γ, as was found to be the case by Korobeinikov (1956). It then follows from (57) and (58) that the dimensionless flow variable distributions will also exhibit this invariance. Solving (59) for β yields

$$\beta = 0{\cdot}429. \tag{60}$$

In terms of Korobeinikov's θ_2, this gives

$$\theta_2 = \beta(1-\beta) = 0{\cdot}245, \tag{61}$$

which compares extremely well with the value of 0·244 from Korobeinikov's exact solution.

To find r as a function of r_0, (57) and (58) can be substituted into the continuity equation (3) and the resulting expression integrated to obtain

$$\frac{r}{R} = \left\{\frac{6\beta(1-2\beta)}{(1-\beta)(3-4\beta)}\ln\left[1+\frac{4\beta^2-7\beta+3}{3-5\beta-4\beta^2}\left(\frac{r_0}{R}\right)^3\right]\right\}^{\frac{1}{3}}. \tag{62}$$

Differentiating this expression with respect to time yields the particle velocity

$$\frac{u}{u_s} = \frac{1}{1-\beta}\left(\frac{r}{R}\right)\left\{1+\frac{6\beta(1-2\beta)}{(1-\beta)(4\beta-3)}\left(\frac{R}{r}\right)^3\left[1-\exp\left\{\frac{(1-\beta)(4\beta-3)}{6\beta(1-2\beta)}\left(\frac{r}{R}\right)^3\right\}\right]\right\}. \tag{63}$$

Substituting (62) into (57) and making use of (58) yields

$$\frac{p}{p_s} = \frac{\rho}{\rho_s} = \frac{3-5\beta-4\beta^2}{6(1-2\beta)}\exp\left\{\frac{(1-\beta)(3-4\beta)}{6\beta(1-2\beta)}\left(\frac{r}{R}\right)^3\right\}. \tag{64}$$

The pressure and density distribution and the velocity distribution given by (64) and (63) above are plotted *versus* the reduced Eulerian co-ordinate r/R in figure 5, and the comparison is seen to be surprisingly good in view of the relatively large value of the shock density ratio β. The 'stagnant core' effect predicted by Korobeinikov (1956) is in evidence in the present approximation, although not to as great an extent. This is not surprising, however, since the present analysis assumes the flow variables to be analytic functions of the Eulerian co-ordinate, and does not admit discontinuous derivatives such as those occurring at $r/R = 0{\cdot}494$ in the exact solution.

In a manner similar to the above, the corresponding flow field distributions can be obtained for $\kappa\to\infty$ but with η finite, i.e. in an exponential atmosphere. However, there are at present no exact solutions with which to compare the results and this has not been carried out.

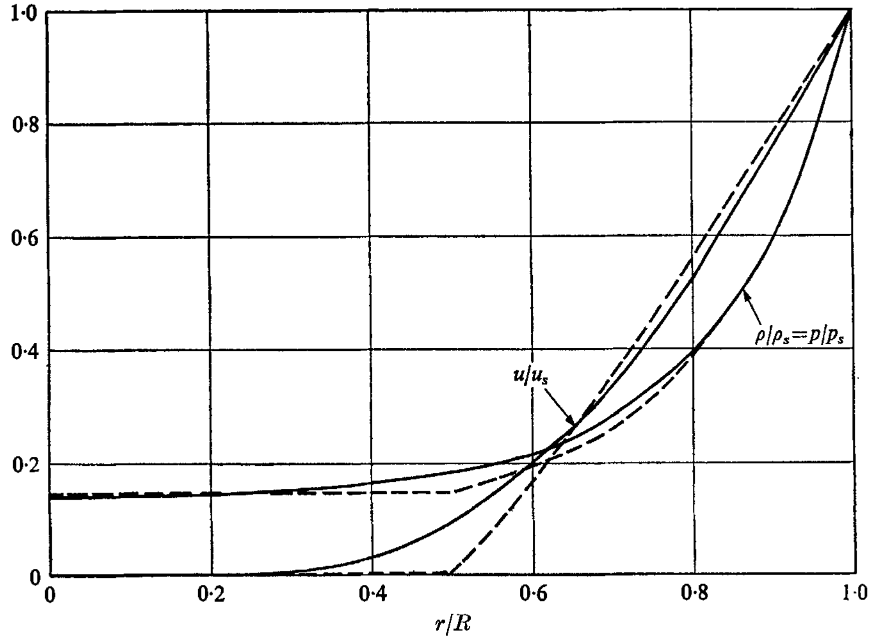

FIGURE 5. Flow variable distributions in Eulerian co-ordinates for uniform density atmosphere in isothermal limit ($\kappa \rightarrow \infty$). ——, present analysis; — — —, Korobeinikov (1956) exact.

5. Results

Numerical results for the exponential atmosphere have been obtained for $\gamma = 1{\cdot}2$ and $\kappa = 0{\cdot}60$. These particular values were selected on the basis of their practical applicability, with the value for γ corresponding to that appropriate for high temperature air and the value for κ corresponding to a burst at about 6 or 7 scale heights above sea level.

The shock velocity $\dot{R}$ in the downward direction is given as a function of the shock position R in figure 6, where it is compared with the results from part 1 for a shock driven by an adiabatic flow field. A significant result of this comparison is that the radiating flow field is preceded by a shock wave travelling at a velocity approximately 30 % faster than that for the adiabatic case. This same result is also found from the uniform density calculations (see (53)). The increased shock speed can be directly attributed to the reduction in the proportion of the total energy exhibited as internal energy when the flow behind the shock is radiating. This is evidenced by the reduction in temperature of the central core of the flow field.

The shock velocity of the upward propagating shock is plotted as a function of the shock position in figure 7 where it is compared with the shock velocity for adiabatic flow. As for the downward propagating shock, there is a significant difference between the shock velocity for the two flow conditions. The higher shock propagation velocity for the radiating flow can once again be attributed to the fact that a smaller proportion of the energy is exhibited as internal energy.

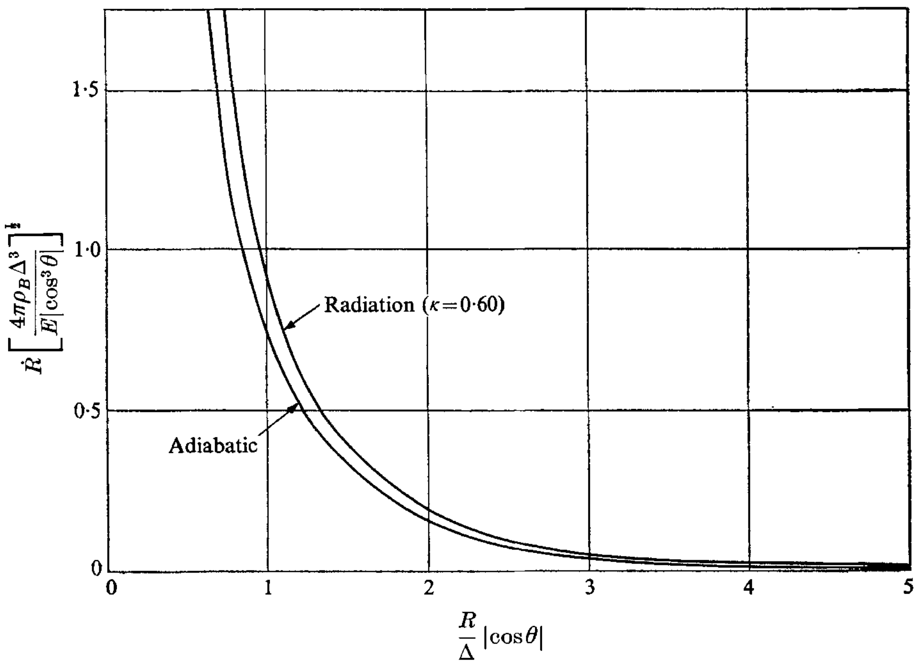

FIGURE 6. Shock velocity in downward direction ($\frac{1}{2}\pi < \theta \leqslant \pi$) as a function of shock position for $\gamma = 1{\cdot}2$.

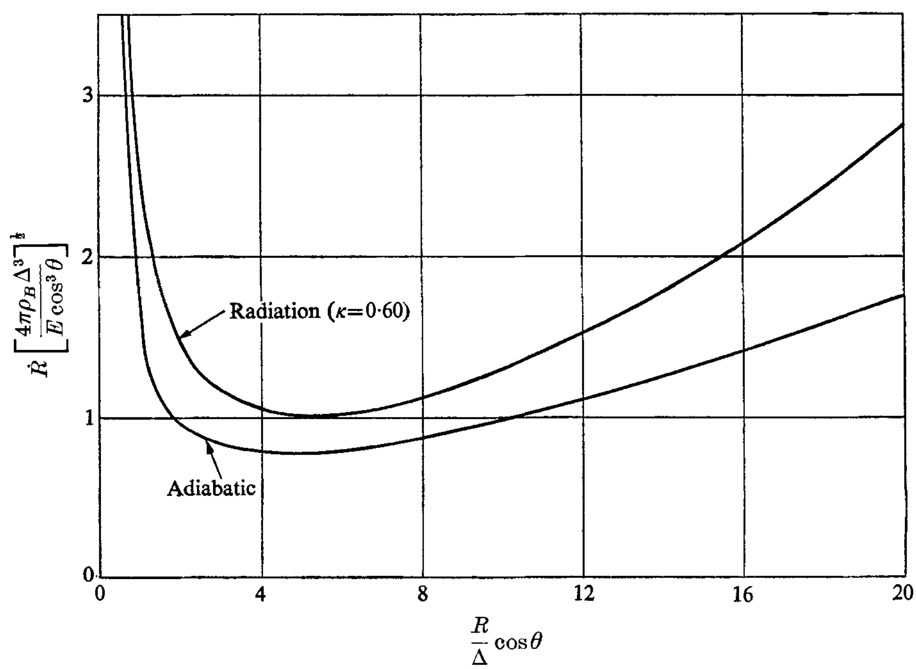

FIGURE 7. Shock velocity in upward direction ($0 \leqslant \theta < \frac{1}{2}\pi$) as a function of shock position for $\gamma = 1{\cdot}2$.

It is seen that the shock displays the same feature as an adiabatic flow, in that it reaches a minimum velocity, beyond which it accelerates to an infinite velocity (within the framework of the non-relativistic approach which has been used).

The flow variables behind the downward propagating shock are shown as a function of r/R in figure 8 for $R\cos\theta/\Delta = -1\cdot0$. It is seen that the pressure and velocity distributions are only slightly affected by the exponential atmosphere, but that an appreciable influence is evidenced in the temperature and density profiles. The temperature profile has steepened considerably from its initial shape and the temperature ratio T_b/T_s has more than doubled. There is, of course, a commensurate change in the density profile. It may also be seen that the flow field for the downward propagating shock is approaching that for adiabatic flow as it must, since $q \rightarrow 0$ as $\eta \rightarrow -\infty$.

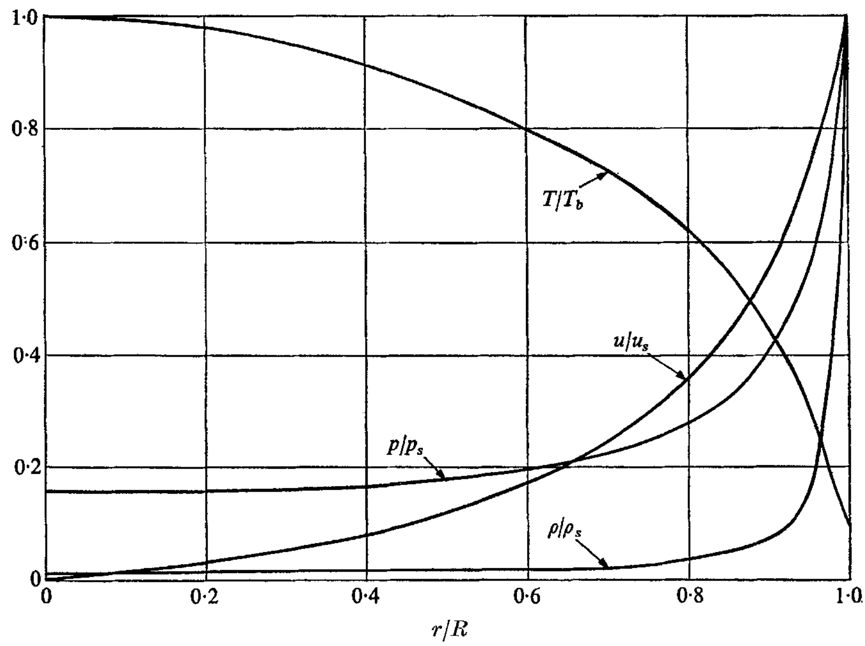

FIGURE 8. Flow variable distributions in Eulerian co-ordinates at $R\cos\theta/\Delta = -1\cdot0$ for $\gamma = 1\cdot2$ and $\kappa = 0\cdot60$; downward direction ($\frac{1}{2}\pi < \theta \leqslant \pi$).

The flow field distribution in Eulerian co-ordinates for the upward propagating shock is given in figure 9 for $R\cos\theta/\Delta = 2\cdot0$. The dimensionless pressure and velocity distributions are seen to be only slightly affected by the exponential atmosphere. However, due to the sensitivity of λ to the exponential density distribution, the temperature profile is appreciably changed, with $T_b/T_s - 1$ reduced from its initial value of 5·4 to 0·35. Correspondingly, the density distribution is seen to be approaching that of the pressure. In figure 10 the flow field for the upward propagating shock is shown at $R\cos\theta/\Delta = 4\cdot0$. Here it is seen that the temperature profile has become essentially flat with the pressure and density curves almost coincident. The velocity and pressure distributions are

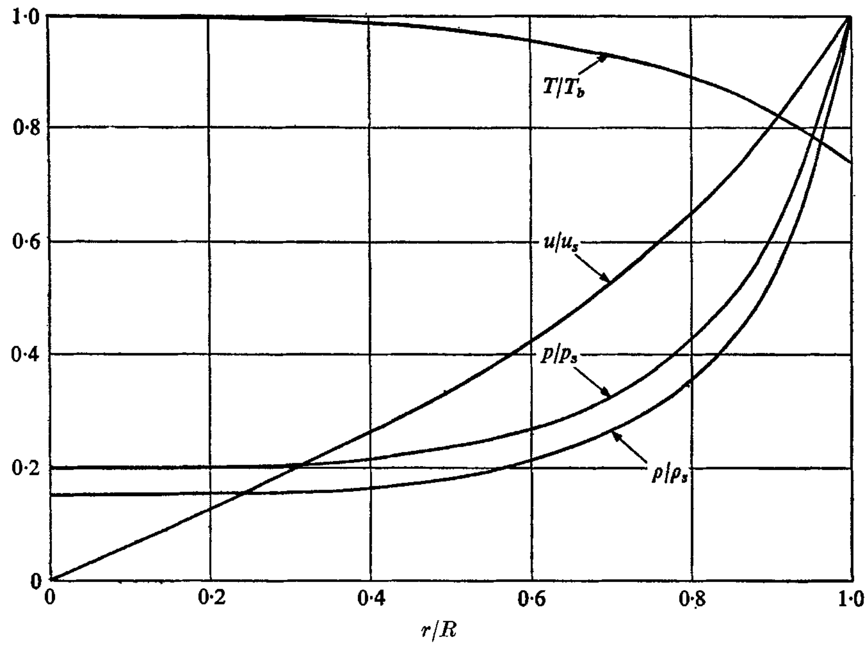

FIGURE 9. Flow variable distributions in Eulerian co-ordinates at $R\cos\theta/\Delta = 2{\cdot}0$ fo $\gamma = 1{\cdot}2$ and $\kappa = 0{\cdot}60$; upward direction ($0 \leqslant \theta < \frac{1}{2}\pi$).

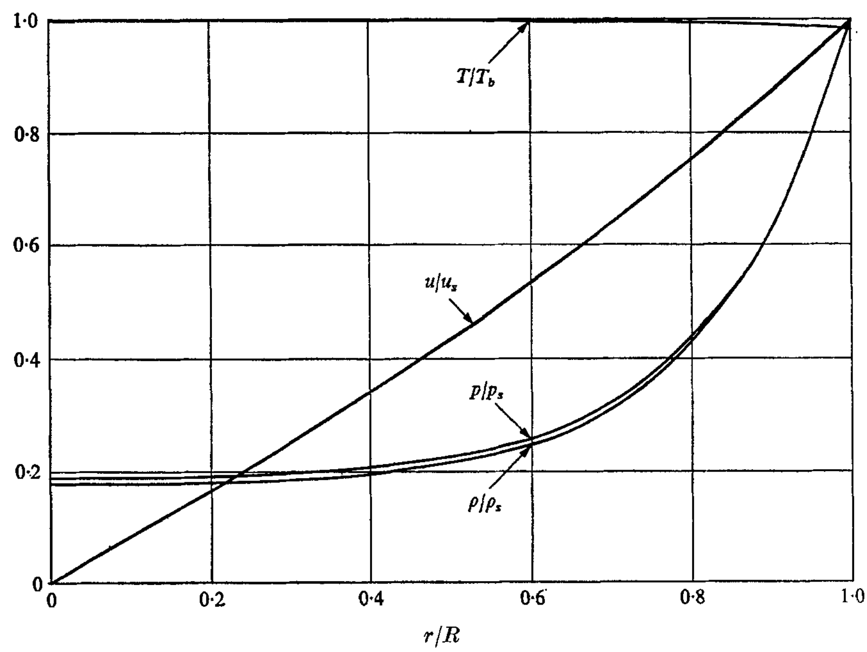

FIGURE 10. Flow variable distributions in Eulerian co-ordinates at $R\cos\theta/\Delta = 4{\cdot}0$ for $\gamma = 1{\cdot}2$ and $\kappa = 0{\cdot}60$; upward direction ($0 \leqslant \theta < \frac{1}{2}\pi$).

seen to be affected only slightly. It should be noted that the flow field distribution is not approaching one characteristic of an adiabatic flow, but instead is approaching that for an isothermal flow. Asymptotically, of course, as $\eta \to \infty$ the Rosseland diffusion approximation is no longer applicable; however, it is quite clear that the temperature profile in the vicinity of the shock front will continue to be flat, owing to the large value of λ and the high temperatures associated with an accelerating shock.

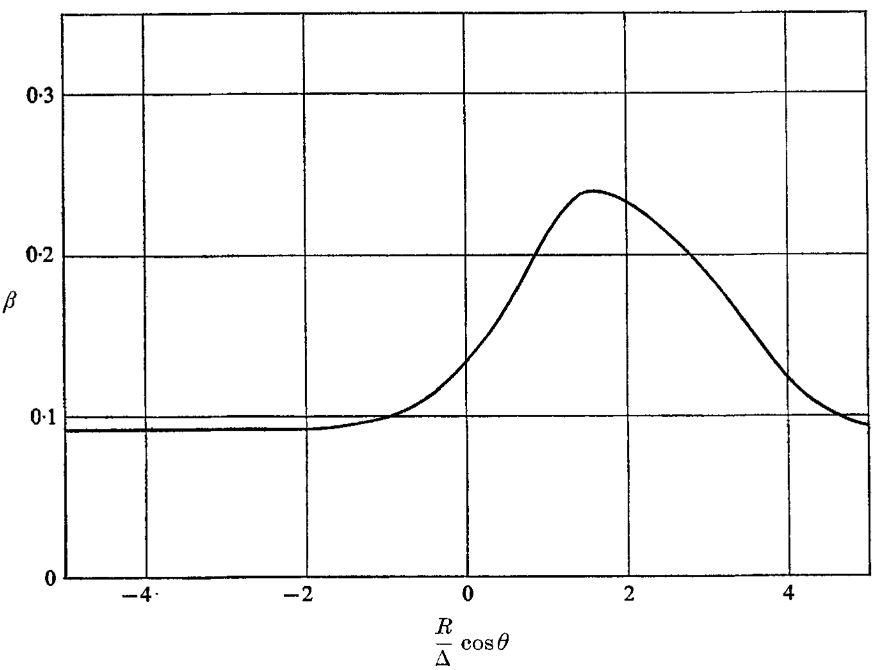

FIGURE 11. Shock density ratio (ρ_0/ρ_s) as a function of shock position for $\gamma = 1{\cdot}2$ and $\kappa = 0{\cdot}60$.

The density ratio across the shock is, of course, a function of time and, consequently, of shock position. The value of the shock density ratio $\beta = \rho_0/\rho_s$ is plotted as a function of shock position in figure 11. For the upward propagating shock, β starts out at the uniform density value of 0·130, and initially increases as the temperature distribution begins to flatten. As the variation of the temperature profile diminishes, the value of β becomes essentially dependent on the pressure distribution, and therefore only on momentum and mass conservation. The approach of β to $(\gamma-1)/(\gamma+1)$ is thus due to the requirements imposed by the simultaneous satisfaction of these two conservation relations. In discussing the shock boundary conditions (9), it has already been pointed out that this result is to be expected for large η.

The value of β for the downward propagating shock diminishes monotonically to $(\gamma-1)/(\gamma+1)$ over a distance of about 1 scale height. The reasons underlying the approach to this limit are significantly different from those for the upward propagating shock. In this case, the heat flux q in (7), evaluated at the shock, approaches zero due to the exponential decrease in the radiation mean free path as seen from (29) and (2). The approach of β to $(\gamma-1)/(\gamma+1)$ in this case was also noted previously when the shock boundary conditions (9) were given.

It is of interest to examine the far field results of the present analysis in the upward and downward directions. For the downward propagating shock, the appropriate asymptotic solution for comparison is the self-similar one obtained by Raizer (1963), since the flow field is approaching the adiabatic limit as $-\eta$ becomes large. In this limit, the third term on the left-hand side of (22) tends to 0/0, since $\beta \to (\gamma-1)/(\gamma+1)$ and $e^{-\eta} \to \infty$. To treat this limit properly, the expansion for $(\partial^2 r/\partial t^2)_s$ in (18*c*) must be appropriately revised for the case where $q = 0$. However, this result has already been obtained in part 1 for adiabatic flow. Consequently, the asymptotic form of (30*b*) is the same as given there. One particular result of interest is the asymptotic expression for the shock velocity which was found to be

$$\dot{R}|\cos\theta| \sim \frac{\alpha\Delta}{t} = \frac{2\Delta}{t}. \tag{65}$$

The descending shock velocity obtained from the integration of (33*b*) and (34) is shown as a function of time in figure 12, where it is compared with the result given by (65) and with the self-similar, asymptotic plane shock solution obtained by Raizer (1963), which is of the same form as (65), but with α a function of γ and different from 2. The value of α was not computed by Raizer for $\gamma = 1{\cdot}2$, however; the value of $\alpha = 1{\cdot}31$, corresponding to the curve attributed to Raizer in figure 12, was obtained from an extrapolation of results for $\alpha(\gamma)$ plotted in figure 10 of part 1. The difference between the two asymptotes can be directly attributed to the difference in the two constraints imposed in obtaining them. The present solution is based upon a conservation of energy constraint, which is appropriate as long as the local radiality assumption is valid, so that the energy contained within a given radial slice is essentially invariant with time. The constraint imposed in the Raizer solution is that the pressure be zero at an infinite distance behind the shock. As noted in part 1, a better comparison between the present analysis and Raizer's solution can be obtained by relaxing the energy constraint and imposing the condition that the pressure go to zero at $r_0 = 0$. It should be emphasized that the appropriate constraint in the régime of practical interest, i.e. distances up to 2–3Δ downward, is the constant energy constraint. Further, it should be noted that the solution does not approach the asymptotic limit until $\eta' \sim 10^{-2}$, or at a distance of about 10Δ, which is well beyond the range of validity of the present analysis.

The asymptotic limit of (30*a*) for η large can be readily obtained (see part 1 for the procedure). In particular, it can be shown that

$$\dot{R}\cos\theta \sim \frac{\alpha\Delta}{\tau-t} = \frac{1-\beta}{\beta}\frac{\Delta}{\tau-t}, \tag{66}$$

where $\beta = (\gamma-1)/(\gamma+1)$, so that $\alpha = 2/(\gamma-1)$. The time τ is the time after burst when $\dot{R}\cos\theta$ becomes infinite and the shock wave emerges at infinity. In view of the fact that the analysis is based upon the Bouguer number being small, an assumption that becomes invalid above 8–10 scale heights, this asymptotic limit has little foundation in physical reality. Nevertheless, it is of interest to note that the shock velocity does have the expected asymptotic form at 'blow-out', though the coefficient α cannot be expected to be reliably given. For this reason, the asymptotic results for the upward propagating shock are not discussed further.

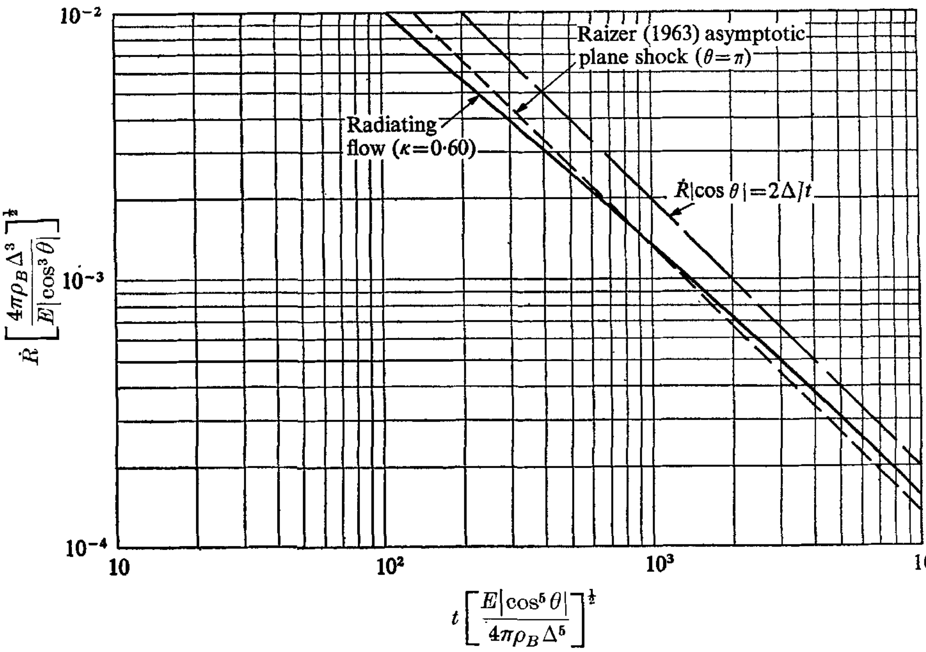

FIGURE 12. Shock velocity in downward direction ($\frac{1}{2}\pi < \theta \leqslant \pi$) as a function of time for $\gamma = 1{\cdot}2$ and $\kappa = 0{\cdot}60$.

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Appendix

Eliminating $\partial^2 r/\partial t^2$ between the radial momentum equation (4) evaluated at the shock and (17*c*) yields

$$-\left.\frac{\partial^2 r}{\partial r_0^2}\right|_R \dot{R}^2 = (1-\beta)\,\dot{R}^2\left(\frac{1}{p}\frac{\partial p}{\partial r_0}\right)_R + (1-\beta)\,\ddot{R} - 2\dot{R}\frac{d\beta}{dt}, \tag{A 1}$$

where the boundary condition (9*b*) has been utilized. Once the pressure gradient term in the above expression has been obtained, the resulting value for $(\partial^2 r/\partial r_0^2)_R$ may be substituted into (17*c*) to obtain an expression for the acceleration in terms of R and its time derivatives.

Logarithmic differentiation of the equation of state (8) yields

$$\frac{1}{p}\frac{\partial p}{\partial r_0}\bigg|_R = \frac{1}{\rho}\frac{\partial \rho}{\partial r_0}\bigg|_R + \frac{1}{T}\frac{\partial T}{\partial r_0}\bigg|_R. \tag{A 2}$$

The first term on the right-hand side can be evaluated by differentiating the conservation of mass relation (3) to obtain

$$\frac{1}{\rho}\frac{\partial \rho}{\partial r_0}\bigg|_R = \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r_0}\bigg|_R + \frac{2}{R}(1-\beta) - \frac{1}{\beta}\frac{\partial^2 r}{\partial r_0^2}\bigg|_R, \tag{A 3}$$

where $(\partial r/\partial r_0)_R$ has been replaced by (16). From (2) it follows that

$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r_0}\bigg|_R = -\frac{\cos\theta}{\Delta}. \tag{A 4}$$

The second term on the right-hand side of (A 2) can be evaluated by making use of the diffusion approximation (7), such that

$$\frac{1}{T}\frac{\partial T}{\partial r}\bigg|_R = -\frac{3}{16}\frac{q_s}{\lambda_s \sigma T_s^4}. \tag{A 5}$$

Making use of the strong shock conditions (9*b*) and (9*c*) along with (16) yields

$$\frac{1}{T}\frac{\partial T}{\partial r_0}\bigg|_R = -\frac{3}{32}\frac{\beta(1-\beta)\rho_0 \dot{R}^3}{\lambda_s \sigma T_s^4}\left[\frac{\gamma(\beta-1)+\beta+1}{\gamma-1}\right]. \tag{A 6}$$

Substituting (A 4) into (A 3) and replacing the pressure gradient term in (A 1) by the resulting expression plus (A 6) determines $(\partial^2 r/\partial r_0^2)_R \dot{R}^2$. Noting that

$$\frac{d}{dt} = \dot{R}\frac{d}{dR}, \tag{A 7}$$

the result given in (18*c*) then follows from (17*c*).

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